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A METHOD OF CONSTRUCTING A WEIGHT FUNCTION FOR A BODY WITH A CRACK†

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A method of constructing a three-dimensional weight function using a variational formula for an elastic body with a crack and the theory of harmonic functions is described. Unlike earlier results [1, 2], with this method it is possible to take into account the variable curvature of the crack contour. The example of a plane embedded elliptical crack is given. © 1998 Elsevier Science Ltd. All rights reserved.

A survey of weight functions is made in [3]. Approximate constructions of weight functions for the general case are given in [4, 5] and applied to elliptical cracks in [6–8].

1. We consider a plane crack in a linearly elastic body which occupies a simply connected volume V, bounded by a surface O. In a rectangular system of coordinates x_1, x_2, x_3, the crack lies in the plane $x_3 = 0$. We shall associate the positive orientation S^+ of the surface of the crack S with $x_3 = 0^+$, and the negative orientation S^- with $x_3 = 0^-$.

The stress intensity factor (SIF) of normal detachment at any point M of the boundary of the crack Γ (the contour of Γ is a two-dimensional curve) can be given by the formula

$$K_1(M) = \iint\limits_{S^+} K_1(M;Q)p(Q)dS, \quad M \in \Gamma, \quad Q \in S^+ \tag{1.1}$$

Thus, if the weight function (WF) $K_1(M; Q)$ is known, the SIF $K_1(M)$ due to arbitrary pressure $p(Q)$ on the surface of the cut S^+ and S^- can be found from formula (1.1).

We will consider the problem of constructing the WF in the general case when the crack contour Γ is a two-dimensional continuously differentiable curve. The method is based on the use of the variational formula [9, 10]

$$\delta_n u_3(Q) = \frac{\pi(1-\nu)}{2\mu} \int\limits_{\Gamma} K_1(M;Q)K_1(M)\delta n(M)ds \tag{1.2}$$

which expresses the variation of the displacement $\delta_n u_3$ of the crack surface S^+ due to variation of the crack contour. Here u_3 is the projection of the displacement vector onto the x_3 axis, ν is Poisson's ratio, μ is the shear modulus and δn is the variation of the crack contour in the direction of the external normal to the curve Γ.

We introduce the notation

$$K_1(M;Q) = K^{1/2}(M)K_1^*(M;Q) \tag{1.3}$$

where $K(M)$ is the curvature of the contour Γ at the point M. From an analysis of known WF, we see that the curvature K appears in the expression for the WF as stipulated in (1.3).

Substituting (1.3) into (1.2), we obtain

$$\delta_n u_3(Q) = \int\limits_{\Gamma} f^*(M)K_1^*(M;Q)ds, \quad f^*(M) = \frac{\pi(1-\nu)}{2\mu} K^{1/2}(M)K_1(M)\delta n(M) \tag{1.4}$$

(the function $f^*(M)$ is given on the curve Γ).

We then use a formula which expresses an harmonic function inside the contour Γ in terms of values on the boundary

$$U(Q) = \int\limits_{\Gamma} f(M)N(M;Q)ds, \quad N(M;Q) = -\frac{\partial G(M;Q)}{\partial n} \tag{1.5}$$

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Here $G(M; Q)$ is Green's function, and the derivative $\partial/\partial n$ is taken along the direction of the outward normal to Γ. Formula (1.5) gives the solution of the problem

$$\Delta U(Q)=0,\quad Q\in S^{+};\quad U|_{\Gamma}=f(M),\quad M\in\Gamma$$

where S^+ is the simply connected region bounded by Γ.

Putting $f^*(M)$ instead of $f(M)$ in (1.5), we obtain

$$U^{*}(Q)=\int_{\Gamma} f^{*}(M)N(M;Q)ds \tag{1.6}$$

Here $U^*(Q)$ is an harmonic function in the region S^+ and $f^*(M)$ are the limiting values of this function on Γ.

We now return to formula (1.4). It can be represented in the form

$$U^{*}(Q)=\int_{\Gamma} f^{*}(M)N_{1}(M;Q)ds,\quad N_{1}(M;Q)=\frac{K_{1}^{*}(M;Q)U^{*}(Q)}{\delta_{n}u_{3}(Q)} \tag{1.7}$$

Applying the Laplace operator to both sides of (1.7), we see that $N_1(M; Q)$ is an harmonic function in the region S^+.

The left-hand sides of expressions (1.6) and (1.7) both contain the harmonic function U^* and have the same integrand. Hence, the harmonic functions N and N_1 must also be equal [11], that is,

$$N(M;Q)=N_{1}(M;Q) \tag{1.8}$$

Substituting expression (1.7) into (1.8) and taking account of relation (1.3), we obtain

$$K_{1}(M;Q)=K^{1/2}(M)N(M;Q)\delta_{n}u_{3}(Q)/U^{*}(Q) \tag{1.9}$$

This is the most general representation for the WF $K_1(M; Q)$.

The functions $K(M)$ and $N(M; Q)$ which appear in (1.9) depend on the crack shape. To find the function $\delta_n u_3/U^*$, which also appears in (1.9) and depends both on the shape of the crack and on the shape of the elastic body, we need to solve some simpler problem for the given body V and a given crack (the trial solution).

A fairly general method of constructing a WF, which is also based on the use of the variational formula (1.2), for a body with a crack was proposed in [1, 2]. However, that method can only be used for cracks with a contour of constant curvature K. Approximate methods of constructing the WF for a crack with an arbitrary contour Γ are described in [4, 5].

If we substitute the expression for the WF (1.9) into the variational formula (1.2), we obtain an identity.

2. We will write formula (1.9) in the form

$$K_{1}(M;Q)=K^{1/2}(M)N(M;Q)F(Q),\qquad F(Q)=\delta_{n}u_{3}(Q)/U^{*}(Q) \tag{2.1}$$

i.e. the function $F(Q)$ should not depend on the form of the "trial solution", since the functions $K_1(M; Q)$, $K(M)$ and $N(M; Q)$ do not depend on that solution either. The easiest way of verifying this fact is to consider the example of an embedded circular crack.

In polar coordinates r, θ, for a circular crack of radius a we have

$$K_{1}(\varphi;r,\theta)=\frac{1}{2\pi a^{1/2}}F(r,\theta)\frac{a^{2}-r^{2}}{A(r,\varphi-\theta)},\quad A(r,\varphi-\theta)=a^{2}+r^{2}-2ar\cos(\varphi-\theta) \tag{2.2}$$

We will consider an infinite elastic isotropic body and show, by means of examples, that the function $F(r, \theta)$ in (2.2) is independent of the "trial solution".

Example 1. A pressure $p(r)$, independent of the angle θ, is applied to the surfaces of the circular crack. Then [12]

$$u_{z}(r,\theta)=\frac{1-\nu}{\mu}\int_{r}^{a}\frac{I(t)dt}{(t^{2}-r^{2})^{1/2}},\quad r\leqslant a;\ I(t)=\frac{2}{\pi}\int_{0}^{t}\frac{\eta p(\eta)d\eta}{(t^{2}-\eta^{2})^{1/2}}$$

Using this expression, we find

$$K_{1}=\frac{1}{a^{1/2}}I(a),\quad \delta_{n}u_{z}(r,\theta)=\frac{1-\nu}{\mu}\frac{a^{1/2}\delta aK_{1}}{(a^{2}-r^{2})^{1/2}}$$

We then obtain expressions for $f^*(\varphi)$ and $U^*(r, \theta)$ from (1.4) and (1.6), and find

$$F(r,\theta)=\frac{\delta_n u_z(r,\theta)}{U^*(r,\theta)}=\frac{2}{\pi(a^2-r^2)^{1/2}} \tag{2.3}$$

Substituting this expression into (2.2), we obtain

$$K_1(\varphi;r,\theta)=\frac{1}{\pi^2 a^{1/2}}\frac{(a^2-r^2)^{1/2}}{A(r,\varphi-\theta)} \tag{2.4}$$

which agrees with known results.

Example 2. We now consider the most general case, in which a pressure $p(r, \theta)$ is applied to the surfaces of the circular crack. In this case the SIF will depend on the angle φ, that is, $K_1 = K_1(\varphi)$. In the previous example we first found the function $F(r, \theta)$ and then the WF $K_1(\varphi; r, \theta)$. However, we can do the opposite and take the expression for the WF (2.4) and use it to find a formula for $F(r, \theta)$, which should be the same as (2.3).

Substituting (2.4) into (1.2), we have an expression for $\delta_n u_z(r, \theta)$. Then, using expressions (1.4) and (1.6), we find $U^*(r, \theta)$. Again we arrive at formula (2.3), and thus the function $F(r, \theta)$ is independent of the form of the "trial solution".

3. We will now use formula (2.1) to find the WF for an embedded elliptical crack in an elastic body. The crack lies in the $x_3 = 0$ plane of the Cartesian system of coordinates x_1, x_2, x_3; a and b are the semi-axes of the bounding ellipse, $a \geqslant b$.

In the $x_3 = 0$ plane we introduce the elliptical coordinates u, υ

$$x_1 = c\,\mathrm{ch}\,u\cos v,\ x_2 = c\,\mathrm{sh}\,u\sin v\quad (u \geqslant 0,\ 0 \leqslant v \leqslant 2\pi)$$

where $c = ae$ and e is the eccentricity of the bounding ellipse. We have

$$K(t) = ab\Pi^{-3/2}(t),\quad \Pi(t) = a^2\sin^2 t + b^2\cos^2 t \tag{3.1}$$

It can be shown that in this case

$$N(M;Q) = N(t;u,v) = \frac{1}{2\pi}\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{\mathrm{ch}\,nu}{\mathrm{ch}\,nu_0}\cos nv\cos nt + \frac{\mathrm{sh}\,nu}{\mathrm{sh}\,nu_0}\sin nv\sin nt\right)\right],\quad \mathrm{th}\,u_0 = k' = \frac{b}{a} \tag{3.2}$$

where u_0 is the bounding ellipse.

Formula (2.1) for an elliptical crack takes the form

$$K_1(t;u,v) = K^{1/2}(t)N(t;u,v)F(u,v) \tag{3.3}$$

The functions $K(t)$ and $N(t; u, \upsilon)$ are defined by expressions (3.1) and (3.2), respectively, and depend only on the crack shape. The function $F(u, \upsilon)$ which appears in (3.3) depends both on the crack shape and on the configuration of the elastic body. We need a "trial solution" in order to find $F(u, \upsilon)$.

We will consider the problem of finding a function $F(u, \upsilon)$ for an infinite elastic body with an embedded elliptical crack. We already have a "trial solution" of this problem in the case where a constant pressure p is applied to the crack surfaces.

In the $x_3 = 0$ plane, we will introduce the generalized polar coordinates ρ and φ

$$x_1 = a\rho\cos\varphi,\quad x_2 = b\rho\sin\varphi\quad (0 \leqslant \rho \leqslant 1,\ 0 \leqslant \varphi \leqslant 2\pi)$$

If $p(\rho, \varphi) = p =$ const, then [12]

$$u_3(\rho,\varphi) = \frac{(1-\nu)bp}{\mu \mathrm{E}(k)}(1-\rho^2)^{1/2},\quad \rho \leqslant 1;\quad K_1(\varphi) = \frac{(k')^{1/2}p}{\mathrm{E}(k)}\Pi^{1/4}(\varphi) \tag{3.4}$$

where $k = (1 - k'^2)^{1/2}$, $k' = b/a$, $\mathrm{E}(k)$ is a complete elliptic integral of the second kind.

From the first formula of (3.4) we have

$$\delta_n u_3(\rho,\varphi) = \frac{(1-\nu)p\delta b}{\mu \mathrm{E}(k)(1-\rho^2)^{1/2}},\quad \frac{\delta b}{\delta a} = k' \tag{3.5}$$

Then, using (1.4), (3.1) and the second formula of (3.4), we find

$$f^*(t)=\frac{\pi(1-\nu)pab\delta b}{2\mu\mathbf{E}(k)}\Pi^{-1}(t) \tag{3.6}$$

(taking $\delta n(t) = b\delta a/\Pi^{1/2}(t)$).

On the contour of the ellipse, that is, when $u = u_0$

$$\sin^2\varphi+(k')^2\cos^2\varphi=\sin^2 t+(k')^2\cos^2 t;\quad d\varphi=dt$$

Substituting (3.6) into (1.6), we obtain

$$U^*(u,v)=\frac{\pi(1-\nu)pab\delta b}{2\mu\mathbf{E}(k)}U_0(u,v),\quad U_0(u,v)=\int_0^{2\pi} N(t;u,v)\Pi^{-1/2}(t)dt \tag{3.7}$$

Then using (3.5) and (3.7) we have

$$F(u,v)=\frac{\delta_n u_3}{U^*}=\frac{2}{\pi ab(1-\rho^2)^{1/2}U_0(u,v)} \tag{3.8}$$

Substituting (3.1) and (3.8) into (3.3), we have finally

$$K_1(t;u,v)=\frac{2N(t;u,v)}{\pi(ab)^{1/2}\Pi^{3/4}(t)(1-\rho^2)^{1/2}U_0(u,v)} \tag{3.9}$$

$$1-\rho^2=1-\left(\frac{k}{k'}\right)^2(\operatorname{sh}^2 u\sin^2 v+k'^2\operatorname{ch}^2 u\cos^2 v)$$

Formula (3.9) defines the WF for an unbounded elastic body with an embedded elliptical crack. If the crack is circular ($a = b$), (3.9) reduces to (2.4).

The WF $K_1(t; u, \upsilon)$ is the SIF of the effect of unit lumped normal forces applied to the crack surfaces at a point with coordinates u, υ.

We now consider the special case of unit lumped forces applied to the crack surfaces at the centre of the ellipse. In this case $u = 0$, $\upsilon = \pi/2$, and formula (3.9) takes the form

$$K_1\left(t;0,\frac{\pi}{2}\right)=\frac{2}{\pi(ab)^{1/2}\Pi^{3/4}(t)U_1}N\left(t;0,\frac{\pi}{2}\right)$$
$$N\left(t;0,\frac{\pi}{2}\right)=\frac{1}{2\pi}\left(1+2\sum_{n=1}^{\infty}\frac{\cos n\pi/2}{\operatorname{ch} nu_0}\cos nt\right),\quad U_1=\int_0^{2\pi}N\left(t;0,\frac{\pi}{2}\right)\Pi^{-1/2}(t)dt \tag{3.10}$$

Suppose further that the elliptical crack is very nearly circular. In that case the semi-axes of the ellipse are equal, respectively, to $(1 + \varepsilon)b$ and $\varepsilon \ll 1$. Then formula (3.10) takes the much simpler form

$$K_1\left(t;0,\frac{\pi}{2}\right)=\frac{1}{\pi^2 b^{3/2}}\left[1-\frac{\varepsilon}{4}(3+5\cos 2t)\right]+O(\varepsilon^2)$$

in agreement with the well-known result [13] obtained by another method.

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